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# MODULAR INVARIANTS OF A QUADRATIC FORM FOR A PRIME POWER MODULUS

A DISSERTATION
SUBMITTED TO THE FACULTY
OF THE OGDEN GRADUATE SCHOOL OF SCIENCE
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
DEPARTMENT OF MATHEMATICS

BY
JAMES ELIJAH McATEE

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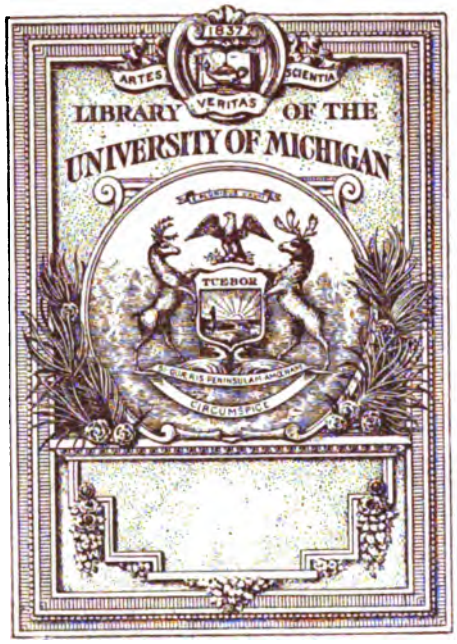
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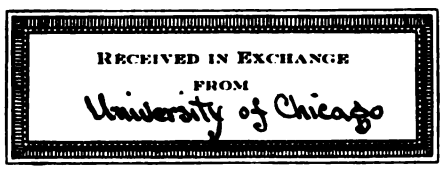
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# *Modular Invariants of a Quadratic Form for a Prime Power Modulus.*

By J. E. McAtee.*

## Introduction.

We consider a quadratic form $q$ in $n$ variables whose coefficients are integers taken modulo $P^\lambda$, where $P$ is an odd prime. Two such forms are said to be equivalent modulo $P^\lambda$ if they become identically congruent under a linear homogeneous transformation whose coefficients are congruent to integers modulo $P^\lambda$ and their determinant is congruent to unity. All the forms equivalent to $q$ are said to form a class. A single-valued function $\phi$ of the coefficients of $q$ is an invariant of $q$ if it has the same value for all forms of the same class. In case the values taken by $\phi$ are integers which may be reduced modulo $P^\lambda$, the invariant is called modular. We obtain a complete invariantive classification of $n$-ary quadratic forms modulo $P^\lambda$ and construct modular invariants which completely characterize the classes. Professor Dickson† had treated the problem for the case $\lambda=1$, when the fundamental invariants are the determinant $D$ of $q$, the rank $r$ modulo $P$ of $D$, and $A_r$, to which the invariant $A_{\bar{R}_0}(D)$ of § 4 reduces for $\lambda=1$.

In his investigation of the number of sets of solutions of the congruence $q\equiv c \pmod{P^\lambda}$, C. Jordan‡ obtained from $q$ by a seriatim process of reduction the canonical form

$$P^{e_1}(A_1x_1^2+\ldots+A_sx_s^2)+P^{e_2}(A_{s+1}x_{s+1}^2+\ldots+A_tx_t^2)+P^{e_3}()+\ldots,$$

but gave no explicit method to deduce à priori from $q$ the number of terms in each parenthesis or the values of $e_1, e_2 \ldots$. This is accomplished by the present invariants.

The second part of the paper is devoted to the polynomial modular invariants of a binary quadratic form modulo $P^\lambda$, in particular to the determination of a fundamental system of such invariants modulo 4.

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* Dr. McAtee died at Urbana, Ill., Dec. 2, 1918. The proof-sheets were compared with both the final and an initial draft of the thesis. In § 7, an inadequate proof of Theorem 1 has been replaced by the present shorter proof, while the easy proofs of some other facts have been omitted. The proof of formula (3) of § 1 has been suppressed, as it was entirely similar to that cited. L. E. Dickson.

† Madison Colloquium Lectures, 1913, Ch. 1; *Trans. Amer. Math. Soc.*, Vol. X (1909), p. 123.

‡ *Jour. de Math.*, (2), Vol. XVII (1872), pp. 368–402. Cf. Bachmann, "Die Arith. der Quadratischen Formen" (1898), p. 434.

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## I. INVARIANTS OF THE $n$-ARY QUADRATIC FORM MODULO $P^\lambda$.

1. *Canonical Forms and Classes.*—Consider the form

$$q_n=\sum_{i,j=1}^{n} a_{ij}x_ix_j \qquad (a_{ij}=a_{ji}), \tag{1}$$

where the $a_{ij}$ are integers taken modulo $P^\lambda$, where $P$ is an odd prime. If every $a_{ij}$ is divisible by $P^\lambda$, the form (1) is said to constitute the class $C_0$. In the contrary case, let $e_1$ be the exponent of the highest power of $P$ that divides every $a_{ij}$, and write

$$q_n=P^{e_1}\sum_{i,j=1}^{n} A_{ij}^{(1)}x_ix_j, \qquad D=|A_{ij}^{(1)}|. \tag{2}$$

First, let the determinant $D$ be prime to $P$. Since there exists a primitive root of $P^\lambda$, it follows as in the Madison Colloquium Lectures, pp. 6, 7, that $q_n$ can be transformed into

$$P^{e_1}[x_1^2+\ldots.+x_{n-1}^2+Dx_n^2] \tag{3}$$

by a linear transformation with integral coefficients of determinant unity modulo $P^\lambda$.

Second, let $D$ be divisible by $P$, and $r_1$ its rank modulo $P$. As on p. 9 of the Lectures cited, we may assume that

$$M_{r_1}=|A_{ij}^{(1)}|\not\equiv 0 \pmod P \qquad (i,j=1,\ldots.,r_1),$$

and show* that $q_n$ can be transformed into

$$P^{e_1}[\sum_{i,j=1}^{n-1} A_{ij}^{(1)}x_ix_j+2\sum_{j=r_1+1}^{n-1} B_{jn}^{(1)}x_jx_n+B_{nn}^{(1)}x_n^2],$$

where

$$B_{jn}^{(1)}\equiv\begin{vmatrix} A_{11}^{(1)}\ldots.A_{1r_1}^{(1)} & A_{1n}^{(1)} \\ \ldots\ldots\ldots\ldots & \\ A_{r_11}^{(1)}\ldots.A_{r_1r_1}^{(1)} & A_{r_1n}^{(1)} \\ A_{j1}^{(1)}\ldots.A_{jr_1}^{(1)} & A_{jn}^{(1)} \end{vmatrix} M_{r_1}^{-1} \qquad (j=r_1+1,\ldots.,n)$$

is congruent to an integer modulo $P^\lambda$ and is divisible by $P$. Repetitions of this process give

$$P^{e_1}[\sum_{i,j=1}^{r_1} A_{ij}^{(1)}x_ix_j+\sum_{i,j=r_1+1}^{n} B_{ij}^{(1)}x_ix_j], \tag{4}$$

where

$$B_{ij}^{(1)}\equiv\begin{vmatrix} A_{11}^{(1)}\ldots.A_{1r_1}^{(1)} & A_{1j}^{(1)} \\ \ldots\ldots\ldots\ldots & \\ A_{r_11}^{(1)}\ldots.A_{r_1r_1}^{(1)} & A_{r_1j}^{(1)} \\ A_{i1}^{(1)}\ldots.A_{ir_1}^{(1)} & A_{ij}^{(1)} \end{vmatrix} M_{r_1}^{-1}$$

is congruent to an integer modulo $P^\lambda$ and is divisible by $P$. If now all the $P^{e_1}B_{ij}^{(1)}$ are zero modulo $P^\lambda$, we have $q_n$ replaced by

$$P^{e_1}\sum_{i,j=1}^{r_1} A_{ij}^{(1)}x_ix_j.$$

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* Using $P^{e_1}B_{jm}\equiv 0 \pmod{P^\lambda}$ in place of $B_{jm}\equiv 0 \pmod P$.

In the contrary case let $e_2$ denote the highest power of $P$ that divides all the $P^{e_1}B_{ij}^{(1)}$ $(e_1<e_2<\lambda)$. Then we have (4) expressed in the form

$$P^{e_1}\sum_{i,\,j=1}^{r_1}A_{ij}^{(1)}x_ix_j+P^{e_2}\sum_{i,\,j=r_1+1}^{n}A_{ij}^{(2)}x_ix_j, \tag{5}$$

where at least one of the $A_{ij}^{(2)}$ is prime to $P$. The question now arises as to whether a different choice of $M_{r_1}$ might not lead to a set of $B_{ij}^{(1)}$ that would give a different $e_2$. To answer this we note that, since $M_{r_1}$ is prime to $P$ and hence is congruent to a power $\rho^m$ of a primitive root $\rho$ of $P^\lambda$, we can transform (5) into

$$P^{e_1}[x_1^2+\ldots.+x_{r_1-1}^2+\rho^m x_{r_1}^2]+P^{e_2}\sum_{i,\,j=r_1+1}^{n}A_{ij}^{(2)}x_ix_j.$$

By applying a transformation of the type

$$x_{r_1}=\rho^{-k}x'_{r_1},\quad x_n=\rho^k x'_n,$$

we obtain one of the forms

$$\left.\begin{aligned}&P^{e_1}[x_1^2+\ldots.+x_{r_1}^2]+P^{e_2}q\quad (q\text{ free of }x_1,\ldots.,x_{r_1}),\\&P^{e_1}[x_1^2+\ldots.+x_{r_1-1}^2+\rho x_{r_1}^2]+P^{e_2}q_\rho\quad (q_\rho\text{ free of }x_1,\ldots.,x_{r_1}).\end{aligned}\right\} \tag{6}$$

If we had chosen $M'_{r_1}$ instead of $M_{r_1}$, we would have obtained from (2) one of

$$\left.\begin{aligned}&P^{e_1}[y_1^2+\ldots.+y_{r_1}^2]+P^{e'_2}q'\quad (q'\text{ free of }y_1,\ldots.,y_{r_1}),\\&P^{e_1}[y_1^2+\ldots.+y_{r_1-1}^2+\rho y_{r_1}^2]+P^{e'_2}q'_\rho\quad (q'_\rho\text{ free of }y_1,\ldots.,y_{r_1}).\end{aligned}\right\} \tag{7}$$

Since * $\qquad x_1^2+\ldots.+x_{r_1}^2$ and $y_1^2+\ldots.+y_{r_1-1}^2+\rho y_{r_1}^2$

are not equivalent modulo $P$, it is evident that if we obtain $(6_1)$ by the use of $M_{r_1}$ we must obtain $(7_1)$ by the use of $M'_{r_1}$. Similarly, if we obtain $(6_2)$ by the use $M_{r_1}$ we must obtain $(7_2)$ by the use of $M'_{r_1}$. For definiteness, suppose that we obtain $(6_1)$ and $(7_1)$ and that $e_2<e'_2$. Then the forms

$$P^{e_1}(x_1^2+\ldots.+x_{r_1}^2)+P^{e_2}q\quad\text{and}\quad P^{e_1}(y_1^2+\ldots.+y_{r_1}^2)$$

are transformable into each other modulo $P^{e'_2}$, whenever the forms

$$x_1^2+\ldots.+x_{r_1}^2+P^kq\quad\text{and}\quad y_1^2+\ldots.+y_{r_1}^2$$

are transformable into each other modulo $P^l$, where $l=e'_2-e_1$ and $k=e_2-e_1$. Let the second of these be transformed into the first by

$$T:\ y_i\equiv\sum_{j=1}^{n}c_{ij}x_j\pmod{P^l}.$$

Employing partial derivatives with respect to $x_i$, we have

$$S:\left\{\begin{aligned}&x_1\equiv c_{11}y_1+\ldots.+c_{r_11}y_{r_1},\\&\ldots\ldots\ldots\ldots\ldots\ldots\\&x_{r_1}\equiv c_{1r_1}y_1+\ldots.+c_{r_1r_1}y_{r_1}\end{aligned}\right\}\pmod{P^l}.$$

* L. E. Dickson, The Madison Colloquium Lectures, 1913, p. 8.

The inverse of $S$ gives $y_i$ $(i=1, \ldots, r_1)$ congruent to linear homogeneous functions of $x_1, \ldots, x_{r_1}$ modulo $P^l$. Hence, if $(6_1)$ and $(7_1)$ are equivalent modulo $P^\lambda$, $P^k q \equiv 0 \pmod{P^l}$ for all integral values of $x_{r_1+1}, \ldots, x_n$. Since at least one of the coefficients in $q$ is prime to $P$, this implies that $k \geqq l$, *i. e.*, that $e_2 \geqq e_2'$. Similarly, $e_2'$ can not be less than $e_2$. Therefore, the choice of the principal minor of $|A_{ij}^{(1)}|$ that is prime to $P$ is immaterial so far as the value of $e_2$ is concerned. Consider now the two forms

$$Q = x_1^2 + \ldots + x_{r_1}^2 + P^l q, \quad Q' = y_1^2 + \ldots + y_{r_1}^2 + P^l q'.$$

Let $Q$ and $Q'$ be equivalent modulo $P^{\lambda - e_1}$ and let

$$T: \; y_i \equiv \sum_{j=1}^{n} c_{ij} x_j \pmod{P^{\lambda - e_1}} \qquad (i=1, \ldots, n)$$

transform $Q'$ into $Q$ modulo $P^{\lambda - e_1}$, where $|c_{ij}| \not\equiv 0 \pmod{P}$. Then we have

$$S: \left\{ \begin{array}{l} \dfrac{\partial Q}{\partial x_1} = 2x_1 \equiv 2c_{11}y_1 + \ldots + 2c_{r_1 1}y_{r_1} + P^l \dfrac{\partial q'}{\partial x_1}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dfrac{\partial Q}{\partial x_{r_1}} = 2x_{r_1} \equiv 2c_{1 r_1}y_1 + \ldots + 2c_{r_1 r_1}y_{r_1} + P^l \dfrac{\partial q'}{\partial x_{r_1}} \end{array} \right\} \pmod{P^{\lambda - e_1}},$$

which shows that $|c_{ij}|$ $(i, j=1, \ldots, r_1)$ is prime to $P$. The inverse of $S$ gives $y_1, \ldots, y_{r_1}$ expressed as linear homogeneous functions of $x_1, \ldots, x_{r_1}$, $y_{r_1+1}, \ldots, y_n$. Moreover, the coefficients of $y_{r_1+1}, \ldots, y_n$ in these functions are divisible by $P^l$. Hence $T$ is expressible in the form

$$y_i \equiv \sum_{j=1}^{r_1} c_{ij}x_j + P^l \sum_{j=r_1+1}^{n} c'_{ij}y_j \pmod{P^{\lambda - e_1}} \qquad (i=1, \ldots, r_1),$$

$$y_i \equiv \sum_{j=1}^{n} c_{ij}x_j \pmod{P^{\lambda - e_1}} \qquad (i=r_1+1, \ldots, n).$$

Subjecting $Q'$ to this transformation, we have

$$\phi(x_1, \ldots, x_{r_1}) + P^l \theta(x_1, \ldots, x_n).$$

Since, by hypothesis, $T$ transforms $Q'$ into $Q$ modulo $P^{\lambda - e_1}$, the terms in $\theta$ involving $x_1, \ldots, x_{r_1}$ must vanish modulo $P^{\lambda - e_1}$, and we have

$$\phi \equiv x_1^2 + \ldots + x_{r_1}^2 \pmod{P^{\lambda - e_1}}.$$

From this we see that the equivalence of $(6_1)$ and $(7_1)$ modulo $P^\lambda$ implies the equivalence of $P^{e_2}q$ and $P^{e_2}q'$. Similarly, for the equivalence of $(6_2)$ and $(7_2)$. This shows that in the method of reduction of a quadratic form to the canonical form (8) or (9) modulo $P^\lambda$, it is immaterial whether we effect it by employing principal minors or by a seriatim process as was done by Jordan in a similar case.* This seriatim process is given in the argument leading up to (3).

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* *Journal de Mathématiques*, Ser. 2, Vol. XVII (1872), pp. 368–375.

Repetitions of the arguments leading to (5) and (6) show that there exists a linear homogeneous transformation of determinant unity modulo $P^\lambda$ under which, when $|P^{-e_1}a_{ij}|$ is divisible by $P$, the form (1) becomes

$$P^{e_1}[x_1^2+\ldots.+x_{r_1-1}^2+\rho^{m_1}x_{r_1}^2]+P^{e_2}[x_{r_1+1}^2+\ldots.+x_{r_1+r_2-1}^2+\rho^{m_2}x_{r_1+r_2}] \\ +\ldots.+P^{e_k}[x_{r_1+r_2+\ldots.+r_{k-1}+1}^2+\ldots.+x_{r_1+\ldots.+r_k-1}^2+\rho^{m_k}x_{r_1+\ldots.+r_k}^2] \quad (8)$$

if $r_1+r_2+\ldots.+r_k<n$, or

$$P^{e_1}[x_1^2+\ldots.+x_{r_1-1}^2+\rho^{m_1}x_{r_1}^2]+P^{e_2}[x_{r_1+1}^2+\ldots.+x_{r_1+r_2-1}^2+\rho^{m_2}x_{r_1+r_2}^2] \\ +\ldots.+P^{e_k}[x_{r_1+\ldots.+r_{k-1}+1}^2+\ldots.+x_{n-1}^2+P^{-\sum_{i=1}^{k}r_ie_i}\rho^{-\sum_{i=1}^{k-1}m_i}Dx_n^2] \quad (9)$$

if $r_1+r_2+\ldots.+r_k=n$, where $m_i=0$ or $1$ [$i=1, \ldots., k$ in (8); $i=1, \ldots., k-1$ in (9)], $D=|a_{ij}|$ and $r_s$ is the rank modulo $P$ of the determinant

$$|A_{ij}^{(s)}| \ (i, j=r_1+r_2+\ldots.+r_{s-1}+1, \ldots., n).$$

$A_{ij}^{(s)}=P^{-e_s}B_{ij}^{(s-1)}$, where the $B_{ij}^{(s-1)}$ are found from the $A_{ij}^{(s-1)}$ in the same way that the $B_{ij}^{(1)}$ were found from $A_{ij}^{(1)}$ in (4). Therefore, when not all the $a_{ij}$ are zero modulo $P^\lambda$, we have the canonical forms (3), (8), (9), and from the work above it is evident that two forms are equivalent modulo $P^\lambda$, if and only if they have the same canonical form. Since (3) is a special case of (9) with $k=1$, *i. e.*, $r_1=n$, we need consider only the canonical forms (8), (9) and the vanishing form where $a_{ij}\equiv 0 \pmod{P^\lambda}$ $(i, j=1, \ldots., n)$. Moreover, since two forms are equivalent modulo $P^\lambda$ if and only if they have the same canonical form, we have the classes $C_0$ and $Ce_im_iR_k$, defined by a canonical form of type (8) with $n>R_k=r_1+\ldots.+r_k$; $Ce_im_jnD$, defined by a canonical form of type (9) with $n=R_k$, for $i=1, \ldots., k$; $j=1, \ldots., k-1$; $e_i<\lambda$.

2. *Elementary Exponents and Ranks.*—Consider the symmetric determinant $D=|a_{ij}|$ $(i, j=1, \ldots., n)$, where the elements are integers. If $P^{\alpha_i}$ is the highest power of $P$ that divides all the $i$-rowed minors of $D$, then $\alpha_i$ is unchanged when $D$ is pre-multiplied or post-multiplied by an $n$-th order determinant whose elements are integers and whose value is unity.* Frobenius called the $P^{e_i}$ $(e_i=\alpha_i-\alpha_{i-1}, e_1=\alpha_1)$ the elementary invariants of $D$ with respect to $P$. We shall call the $e_i$ the elementary exponents of $D$ with respect to $P$, or simply the elementary exponents of $D$. Note for later use that† $e_i$ is a monotonically increasing function of $i$. Let $P^{e_1}$ be the highest power of $P$ that divides all the $a_{ij}$ and consider the case when $e_1<\lambda$. Let the rank modulo $P$ of

* Stephen Smith, *Phil. Trans.*, Vol. CLI, § 12, p. 311. Bachmann's *Zahlentheorie*, Vol. IV. Frobenius, *Berlin Sitzungsberichte* (1894), Pt. I, p. 31. These properties hold if $D$ is not symmetric, but our interest lies in the theorem as given.

† Frobenius, *loc. cit.* Also seen from this section.

the determinant $|A_{ij}| = |P^{-e_1}a_{ij}|$ be $r_1$, $0<r_1<n$. Then there exists* a principal minor $M_{r_1}$ of $|A_{ij}|$ of order $r_1$ that is prime to $P$. By multiplying $D$ by properly chosen unit determinants we may assume that

$$|A_{ij}| = M_{r_1} \not\equiv 0 \pmod{P} \qquad (i, j = 1, \ldots, r_1).$$

Consider the unit $n$-rowed determinant $d^{(1)} = |d_{st}^{(1)}|$ in which each element is zero except

$$d_{st}^{(1)} = 1 \text{ when } s = t, \quad d_{r_1+1,\,t}^{(1)} \equiv -\frac{A_{1,\,r_1+1}\bar{A}_{1t}}{M_{r_1}} \pmod{P^\lambda} \qquad (t = 1, \ldots, r_1),$$

where $\bar{A}_{ij}$ denotes the cofactor of $A_{ij}$ in $M_{r_1}$. If we pre-multiply $D$ by $d^{(1)}$ we have, upon denoting the resulting elements by $b_{ij}^{(1)}$,

$$b_{1,\,r_1+1}^{(1)} \equiv a_{1,\,r_1+1} - \frac{a_{1,\,r_1+1}}{M_{r_1}} \sum_{i=1}^{r_1} A_{1i}\bar{A}_{1i} \equiv 0 \pmod{P^\lambda},$$

$$b_{k,\,r_1+1}^{(1)} \equiv a_{k,\,r_1+1} - \frac{a_{1,\,r_1+1}}{M_{r_1}} \sum_{i=1}^{r_1} A_{ki}\bar{A}_{1i} \pmod{P^\lambda} \qquad (k \neq 1).$$

Hence

$$b_{k,\,r_1+1}^{(1)} \equiv a_{k,\,r_1+1} \pmod{P^\lambda} \qquad (1 < k \leq r_1),$$

$$b_{k,\,r_1+1}^{(1)} \equiv \frac{P^{e_1}}{M_{r_1}} [A_{k,\,r_1+1}M_{r_1} - A_{1,\,r_1+1}M_{r_1 1}^{(k)}] \qquad (k > r_1),$$

where $M_{r_1 1}^{(k)}$ denotes the determinant formed from $M_{r_1}$ by replacing its first row or column by the corresponding elements of the $k$-th row or column of $|A_{ij}|$ $(i, j = 1, \ldots, n)$. All other elements of $D$ are unchanged. If we now pre-multiply this resulting determinant by $d^{(2)} = |d_{st}^{(2)}|$, where $s, t = 1, \ldots, n$, and every element is zero except

$$d_{st}^{(2)} = 1 \text{ when } s = t; \quad d_{r_1+1,\,t}^{(2)} \equiv -\frac{A_{2,\,r_1+1}\bar{A}_{2t}}{M_{r_1}} \qquad (t = 1, \ldots, r_1),$$

we have

$$b_{1,\,r_1+1}^{(2)} \equiv b_{2,\,r_1+1}^{(2)} \equiv 0 \pmod{P^\lambda}, \quad b_{k,\,r_1+1}^{(2)} \equiv a_{k,\,r_1+1} \qquad (k = 3, 4, \ldots, r_1),$$

$$b_{k,\,r_1+1}^{(2)} \equiv \frac{P^{e_1}}{M_{r_1}} [A_{k,\,r_1+1}M_{r_1} - A_{1,\,r_1+1}M_{r_1 1}^{(k)} - A_{2,\,r_1+1}M_{r_1 2}^{(k)}] \pmod{P} \quad (k > r_1).$$

All other elements are again unchanged. Continuing this process, we ultimately obtain from $D$ a determinant having its first $r_1$ elements in the $(r_1+1)$-th row zero modulo $P^\lambda$ and the $k$-th element of this row $(k > r_1)$ is

$$P^{e_1}B_{k,\,r_1+1}^{(1)} \equiv \begin{vmatrix} A_{11} & \ldots . A_{1r_1} & A_{1k} \\ \ldots & \ldots\ldots & \ldots \\ A_{r_1 1} & \ldots . A_{r_1 r_1} & A_{r_1 k} \\ A_{r_1+1,\,1} & \ldots . A_{r_1+1,\,r_1} & A_{r_1+1,\,k} \end{vmatrix} P^{e_1}M_{r_1}^{-1} \pmod{P^\lambda}.$$

* L. E. Dickson, *Annals of Mathematics*, Ser. 2, Vol. XV (1913), pp. 27, 28.

$B^{(1)}_{k,r_1+1}$ is divisible by $P(r_1<k\leqq n)$, since $r_1$ is the rank modulo $P$ of the determinant $|A_{ij}|$ $(i, j=1, \ldots, n)$. All the elements of $D$ in any other row are unchanged. Operating similarly upon the $(r_1+2)$-th, ...., $n$-th row of this determinant and then post-multiplying by a properly chosen unit determinant, we ultimately obtain from $D$

$$D^{(1)}\equiv\begin{vmatrix} A & O \\ O & B \end{vmatrix} \pmod{P^\lambda}, \quad A=\begin{pmatrix} a_{11} \ldots . a_{1r_1} \\ \ldots\ldots\ldots \\ a_{r_11} \ldots . a_{r_1r_1} \end{pmatrix}, \quad B=\begin{pmatrix} P^{e_1}B^{(1)}_{r_1+1r_1+1} \ldots . P^{e_1}B^{(1)}_{r_1+1n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ P^{e_1}B^{(1)}_{nr_1+1} \ldots . P^{e_1}B^{(1)}_{nn} \end{pmatrix},$$

where $O$ is a matrix all of whose elements are zero.

Here $B^{(1)}_{ij}$ $(i, j=r_1+1, \ldots, n)$ is divisible by $P$. To determine the highest power of $P$ that divides all the $P^{e_1}B^{(1)}_{ij}$, let $P^{e'_2}$ denote the highest power of $P$ that divides all the $(r_1+1)$-rowed minors of $D$. Then $P^{e'_2}$ is the highest power of $P$ that divides all the $(r_1+1)$-rowed minors of $D^{(1)}$. Then evidently $P^{e'_2-e_1r_1}$ is the highest power of $P$ that divides all the $P^{e_1}B^{(1)}_{ij}$ of $D^{(1)}$. Let $e_2=e'_2-r_1e_1$. Then $e_2$ is the $(r_1+1)$-th elementary exponent of $D$. We shall call $r_1$ the first elementary rank of $D$ with respect to $P$. Consider the case when $e_2<\lambda$. Let $r_2$ be the rank modulo $P$ of the determinant

$$|P^{-(e_2-e_1)}B^{(1)}_{ij}| \qquad (i, j=r_1+1, \ldots . n).$$

Since there is a principal minor of this determinant of order $r_2$ that is prime to $P$, we may operate on this determinant as we did on $D$ above. Moreover, considering this determinant as it stands in $D^{(1)}$, we see that this operation would leave the first $r_1$ rows and columns of $D^{(1)}$ unaltered. Hence we would have

$$D^{(2)}\equiv\begin{vmatrix} A & O & O \\ O & B^{(1)} & O \\ O & O & C \end{vmatrix} \pmod{P^\lambda},$$

$$B^{(1)}=\begin{pmatrix} P^{e_1}B^{(1)}_{r_1+1, r_1+1} \ldots . P^{e_1}B^{(1)}_{r_1+1, r_1+r_2} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ P^{e_1}B^{(1)}_{r_1+r_2, r_1+1} \ldots . P^{e_1}B^{(1)}_{r_1+r_2, r_1+r_2} \end{pmatrix}, \quad C=\begin{pmatrix} P^{e_3}B^{(2)}_{r_1+r_2+1, r_1+r_2+1} \ldots . P^{e_3}B^{(2)}_{r_1+r_2+1, n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ P^{e_3}B^{(2)}_{n, r_1+r_2+1} \ldots\ldots . P^{e_3}B^{(2)}_{n, n} \end{pmatrix}.$$

Let $P^{e'_3}$ be the highest power of $P$ that divides all the $(r_1+r_2+1)$-rowed minors of $D$. Then $e'_3-r_1e_1-r_2e_2=e_3$ is the highest power of $P$ that divides all the $P^{e_3}B^{(2)}_{ij}$ of $D^{(2)}$. Evidently we may continue this process until all the new elements are zero modulo $P^\lambda$, or until $\Sigma r_i=n$. At any stage, say the $t$-th, $e_t$ and $r_t$ are determined by the preceding. In fact

$$e_t=e'_t-\sum_{i=1}^{t-1} r_ie_i,$$

where $e'_t$ is the highest power of $P$ that divides all the $(r_1+r_2+\ldots .+r_{t-1}+1)$-rowed minors of $D$, and $r_t$ is an integer such that $E=r_1e_1+\ldots .+r_te_t$ is the

exponent of the highest power of $P$ that divides all the $(r_1+r_2+\ldots+r_i)$-rowed minors of $D$ while all the $(r_1+\ldots+r_i+1)$-rowed minors are divisible by a higher power of $P$ than $P^{E+e_i}$. We shall refer to these $r_i$ as the elementary ranks of $D$ with respect to $P$.

*Definition.* The determinant $D$ shall be said to be of rank $R$ modulo $P^\lambda$ if the $R$-th elementary exponent of $D$ with respect to $P$ is less than $\lambda$, while the $(R+1)$-th elementary exponent of $D$ is $\geqq\lambda$.

From this definition it follows that the rank of $D$ modulo $P^\lambda$ is an invariant under multiplication of $D$ by unit determinants and that under such multiplication $D$ becomes

$$D^{(k)}\equiv\begin{vmatrix} A & 0 & 0\ldots.0 & 0 & 0\ldots. \\ 0 & B^{(1)} & 0\ldots.0 & 0 & 0\ldots. \\ \ldots\ldots\ldots\ldots\ldots\ldots & & & & \\ 0 & 0 & 0\ldots.0 & B^{(k)} & 0\ldots. \\ \ldots\ldots\ldots\ldots\ldots\ldots & & & & \end{vmatrix} \pmod{P^\lambda},$$

$$B^{(k)}=\begin{pmatrix} P^{e_k}B^{(k)}_{R_k+1,\,R_k+1}\ldots.P^{e_k}B^{(k)}_{R_k+1,\,R_{k+1}} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ P^{e_k}B^{(k)}_{R_{k+1},\,R_k+1}\ldots.P^{e_k}B^{(k)}_{R_{k+1},\,R_{k+1}} \end{pmatrix},$$

where $R_i=r_1+\ldots+r_i$, and $R_k$ is the rank of $D$ modulo $P^\lambda$. The $e_1, e_2, \ldots, e_k$ are the distinct elementary exponents of $D$ that are less than $\lambda$ arranged in the order of increasing magnitude. From these considerations it follows that if the rank modulo $P^\lambda$ of a symmetric determinant $D$ is $R>0$, there exists a principal minor $M_R$ of $D$ of order $R$ whose rank modulo $P^\lambda$ is $R$, and such that the highest power of $P$ that divides all the $R$-rowed minors of $D$ is exactly the power of $P$ that is contained in $M_R$, or, more precisely, whose elementary exponents are those of $D$ that are less than $\lambda$ and whose elementary ranks are precisely those of $D$. In fact, the $M_{R_k}$ formed by taking the first $R_k$ rows and columns of $D^{(k)}$ is congruent modulo $P^\lambda$ to such a principal minor of $D$ of order $R_k$. From the form of $D^{(k)}$ also follows the fact that the elementary exponent is a monotonically increasing function of the order of the minors.

3. *Criteria for the Classes $Ce_im_iR_k$ and $Ce_im_jnD$.*—It follows from §§ 1 and 2 that the determinant $D$ of a quadratic form, the $e_i$, $r_i$ of $D$ and the rank $R_k=r_1+\ldots+r_k$ of $D$ modulo $P^\lambda$ are invariants. However, these invariants do not completely characterize the classes. We seek criteria for the determination of the $m_i$ of (8) and (9) § 1. Consider the modular form $q_n$ given by (1) and let not all the $a_{ij}$ be zero modulo $P^\lambda$. Under transformations of the type used in § 1, $q_n$ becomes

$$\sum_{s=1}^{k} P^{e_s} \sum_{i,\,j=R_{s-1}+1}^{R_s} A^{(s)}_{ij}x_ix_j, \quad R_s=\sum_{i=1}^{s} r_i, \tag{10}$$

where $e_1<e_2<\ldots.<e_k$ are the distinct elementary exponents of $D$ with respect to $P$ that are less than $\lambda$, and the $r_i$ are the elementary ranks of $D$ with respect to $P$. The determinants $|A_{ij}^{(s)}|$ are prime to $P$, and $R_k$ is the rank modulo $P^\lambda$ of $D$.

Consider first the case $R_k<n$. Let $E_i=r_1e_1+\ldots.+r_ie_i$ and denote the minor $|P^{e_1}A_{ij}^{(1)}|$ of the determinant of (10) by $M_{R_1}$. Then $q_n$ belongs to the class $Ce_im_iR_k$ with $m_1=0$ or 1, according as $(P^{-E_1}M_{R_1})^{\frac{\mu}{2}}\equiv 1$ or $-1 \pmod{P^\lambda}$, where $\mu=\phi(P^\lambda)=P^{\lambda-1}(P-1)$. By §§ 1, 2, $M_{R_1}$ is congruent modulo $P^\lambda$ to an $R_1$-rowed principal minor of $D$ such that $P^{-E_1}M_{R_1}$ is prime to $P$. Let now $M_{R_1}$ denote any such principal minor of $D$. Then, since $q_n$ has one and but one of the canonical forms, we conclude that $q_n$ belongs to the class $Ce_im_iR_k$ with $m_1=0$ or 1 according as $(P^{-E_1}M_{R_1})^{\frac{\mu}{2}}\equiv 1$ or $-1 \pmod{P^\lambda}$. Consider next the minor

$$M_{R_2}\equiv\begin{vmatrix} P^{e_1}A_{11}^{(1)} \ldots. P^{e_1}A_{1,R_1}^{(1)} & 0\ldots\ldots\ldots\ldots 0 \\ \ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ P^{e_1}A_{R_1,1}^{(1)}\ldots. P^{e_1}A_{R_1R_1}^{(1)} & 0\ldots\ldots\ldots\ldots 0 \\ 0\ldots\ldots\ldots 0\ldots\ldots & P^{e_2}A_{R_1+1,R_1+1}^{(2)}\ldots. P^{e_2}A_{R_1+1,R_2}^{(2)} \\ \ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ 0\ldots\ldots\ldots 0\ldots\ldots & P^{e_2}A_{R_2,R_1+1}^{(2)}\ldots\ldots P^{e_2}A_{R_2R_2}^{(2)} \end{vmatrix}.$$

$M_{R_2}$ is an $R_2$-rowed principal minor of the determinant of (10) and is congruent modulo $P^\lambda$ to an $R_2$-rowed principal minor of $D$. Moreover, by §§ 1, 2, $q_n$ belongs to the class $Ce_im_iR_k$ with $m_2=0$ or 1 according as

$$(|A_{ij}^{(2)}|)^{\frac{\mu}{2}}=\left(\frac{P^{-E_2}M_{R_2}}{|A_{ij}^{(1)}|}\right)^{\frac{\mu}{2}}\equiv(|A_{ij}^{(1)}|)^{\frac{\mu}{2}}(P^{-E_2}M_{R_2})^{\frac{\mu}{2}}\equiv 1 \text{ or } -1 \pmod{P^\lambda}.$$

Let now $M_{R_2}$ denote any $R_2$-rowed principal minor of $D$ having an $R_1$-rowed principal minor $M_{R_1}$ such that $P^{-E_2}M_{R_2}$ and $P^{-E_1}M_{R_1}$ are prime to $P$. Then, in view of §§ 1, 2, and the fact that $q_n$ has one and but one of the canonical forms, $q_n$ belongs to the class $Ce_im_iR_k$ with $m_2=0$ or 1 according as

$$(P^{-E_1}M_{R_1})^{\frac{\mu}{2}}(P^{-E_2}M_{R_2})^{\frac{\mu}{2}}\equiv 1 \text{ or } -1 \pmod{P^\lambda}.$$

In general let $M_{R_s}$ be any $R_s$-rowed principal minor of $D$ having an $R_{s-1}$-rowed principal minor $M_{R_{s-1}}$, which in turn has an $R_{s-2}$-rowed principal minor $M_{R_{s-2}}$ and so on to $M_{R_2}$ which has an $R_1$-rowed principal minor $M_{R_1}$ such that $P^{-E_i}M_{R_i}$ $(i=1,\ldots.,s)$ is prime to $P$. Then by a transformation of determinant unity which merely permutes the $x_i$ of (1) we have the principal minors $M_{R_1}$ occupying the first $R_1$ rows and columns of $D$, etc. Then from the form of $D^{(k)}$ of § 2 we have that $q_n$ belongs to the class $Ce_im_iR_k$ with $m_s=0$ or 1 according as

$$\prod_{j=1}^{s}(P^{-E_j}M_{R_j})^{\frac{\mu}{2}}\equiv 1 \text{ or } -1 \pmod{P^\lambda}. \qquad (11)$$

For the case $R_k=n$, $D$ together with (11) for $s=1, \ldots, k-1$ furnish criteria. At the first consideration one would probably be surprised that $D$ may be zero modulo $P^\lambda$ and still $q_n$ belong to class $Ce_i m_j nD$. In fact $D$ may be divisible by a much higher power of $P$ than $P^\lambda$ and still $q_n$ belong to class $Ce_i m_j nD$, since it is not the power of $P$ that divides the minors of $D$ that determine the number of terms in the canonical form, but the elementary exponents of $D$. This phenomenon is what gave rise to the definition of the rank of $D$ modulo $P^\lambda$ of § 2. The rank of $D$ modulo $P^\lambda$ shall be called the rank of the form $q_n$ modulo $P^\lambda$. Two forms $q_n$ and $q'_n$ of rank $n$ are equivalent if and only if

$$e_i=e'_i,\ r_i=r'_i,\ m_j=m'_j \qquad (i=1, \ldots, k;\ j=1, \ldots, k-1)$$

and $P^{-E_k+e_k}D \equiv P^{-E'_k+e'_k}D'$ (mod $P^\lambda$) as is seen by §§ 1, 2.

4. *Invariantive Criteria for the Classes.*—Consider the modular quadratic form (1), and let $e_1<e_2<\ldots<e_h$ be the distinct elementary exponents of $D$ with respect to $P$. Let $\overline{D}$, $\bar{e}_i$, $\bar{r}_i$ and $\bar{m}_i$ be a consistent set of values that define a class $C\bar{e}_i\bar{m}_i\bar{R}_k$ or $C\bar{e}_i\bar{m}_j n\overline{D}$ according as the rank $\bar{R}_k$ modulo $P^\lambda$ of $\overline{D}$ is less than or equal to $n$. For abbreviation set

$$\begin{aligned}
&\mathfrak{R}_i=\sum_{j=1}^{i-1} r_j+\bar{r}_i, \quad \mathfrak{E}_i=\sum_{j=1}^{i-1} r_je_j+\bar{r}_ie_i && (i=1, \ldots, k),\\
&\bar{R}_i=\sum_{j=1}^{i} \bar{r}_j, \qquad \bar{E}_i=\sum_{j=1}^{i} \bar{r}_j\bar{e}_j && (i=1, \ldots, k),\\
&R'_i=\sum_{j=1}^{i-1} r_j+r'_i, \quad E'_i=\sum_{j=1}^{i-1} r_je_j+r'_ie_i && (i=1, \ldots, k;\ \bar{r}_i<r'_i\leqq n-R_{i-1}),\\
&R_i=\sum_{j=1}^{i} r_j, \qquad E_i=\sum_{j=1}^{i} r_je_j && (i=1, \ldots, h).
\end{aligned}$$

Thus $R_h$ and $\bar{R}_k$ are the ranks modulo $P^\lambda$ of $D$ and $\overline{D}$, respectively.

Consider the function*

$$\begin{aligned}
A_{\bar{R}_1}(D) \equiv (P^{e_1-\bar{r}_1})^\mu \{ &(P^{-\mathfrak{E}_1}M^{(1)}_{\mathfrak{R}_1})^{\frac{\mu}{2}}+(P^{-\mathfrak{E}_1}M^{(2)}_{\mathfrak{R}_1})^{\frac{\mu}{2}}[1-(P^{-\mathfrak{E}_1}M^{(1)}_{\mathfrak{R}_1})^\mu]\\
&+\ldots+(P^{-\mathfrak{E}_1}M^{(r)}_{\mathfrak{R}_1})^{\frac{\mu}{2}}\prod_{j=1}^{r-1}[1-(P^{-\mathfrak{E}_1}M^{(j)}_{\mathfrak{R}_1})^\mu]\}\prod_i[1-(P^{-E'_1}M^{(i)}_{R'_1})^\mu] \pmod{P^\lambda}, \quad (12)
\end{aligned}$$

where $M^{(j)}_{\mathfrak{R}_1}$ ranges over the principal minors of $D$ of order $\mathfrak{R}_1=\bar{R}_1$, and $M^{(j)}_{R'_1}$ ranges over the principal minors of $D$ of order $R'_1>\mathfrak{R}_1$. Since $P^{e_1}$ divides all the $a_{ij}$ of $D$, it is evident that there is no term of $A_{\bar{R}_1}(D)$ that has as factor $P$ with a negative exponent. $A_{\bar{R}_1}(D)$ is zero modulo $P^\lambda$, unless $e_1=\bar{e}_1$ and unless at least one of the $P^{-\mathfrak{E}_1}M_{\mathfrak{R}_1}$ is prime to $P$ (since $P\geqq 3$). This implies that $A_{\bar{R}_1}(D)\equiv 0 \pmod{P^\lambda}$ unless the rank modulo $P$ of the determinant $|P^{-e_1}a_{ij}|$ is $\bar{r}_1$. In fact, if $r_1<\bar{r}_1$, none of the $P^{-\mathfrak{E}_1}M_{\mathfrak{R}_1}$ are prime to $P$, while if $r_1>\bar{r}_1$, at least one

* In case $k=1$ and $\lambda>1$, $A_{\bar{R}_1}(D)$ is to have also the final factor of (15).

of the $P^{-E'_1}M_{R'_1}$ is prime to $P$ and the final factor is zero by Fermat's theorem. Thus the value of $A_{\bar{R}_1}(D)$ is zero modulo $P^\lambda$, unless $q_n$ belongs to a class $Ce_im_iR_k$ or $Ce_im_inD$ with $e_1=\bar{e}_1$ and $r_1=\bar{r}_1$, and by (11), § 3, its value modulo $P^\lambda$ is 1 or $-1$ according as $m_1=0$ or 1.

Consider next the function

$$A_{\bar{R}_2}(D)\equiv(P^{e_2-r_2})^\mu\{A_{\bar{R}_1}(M^{(1)}_{\mathfrak{R}_2})\,(P^{-\mathfrak{G}_2}M^{(1)}_{\mathfrak{R}_2})^{\frac{\mu}{2}}+A_{\bar{R}_1}(M^{(2)}_{\mathfrak{R}_2})\,(P^{-\mathfrak{G}_2}M^{(2)}_{\mathfrak{R}_2})^{\frac{\mu}{2}}[1-(P^{-\mathfrak{G}_2}M^{(1)}_{\mathfrak{R}_2})^\mu]$$
$$+\ldots.+A_{\bar{R}_1}(M^{(\sigma)}_{\mathfrak{R}_2})(P^{-\mathfrak{G}_2}M^{(\sigma)}_{\mathfrak{R}_2})^{\frac{\mu}{2}}\prod_{j=1}^{s-1}[1-(P^{-\mathfrak{G}_2}M^{(j)}_{\mathfrak{R}_2})^\mu]\}\prod_{i}[1-P^{-E'_2}M^{(i)}_{R'_2})^\mu]\ (\mathrm{mod}\,P^\lambda),\quad(13)$$

where $M^{(j)}_{\mathfrak{R}_2}$ ranges over the principal minors of $D$ of order $\mathfrak{R}_2$, and $M^{(i)}_{R'_2}$ ranges over the principal minors of $D$ of orders $R'_2>\mathfrak{R}_2$. That the principal minors are divisible by the power of $P$ indicated in (13) follows from the fact that the elementary exponents of $D$ increase monotonically with the order of the minors to which they correspond. In fact, by hypothesis, every minor of order $R_1$ of $D$ is divisible by $P^{r_1e_1}$ while every minor of order $R_1+1$ is divisible by $P^{r_1e_1+e_2}$. Let $\varepsilon$ be the $(R_1+2)$-th elementary exponent of $D$. Then every minor of $D$ of order $R_1+2$ is divisible by $P^{r_1e_1+e_2+\epsilon}$. From the fact that $\varepsilon\geqq e_2$ we see that every minor of $D$ of order $R_1+2$ is divisible by $P^{r_1e_1+2e_2}$. Continuing this we have that every minor of $D$ of order $R_1+\bar{r}_2=\mathfrak{R}_2$ is divisible by $P^{r_1e+r_2e_2}=P^{\mathfrak{G}_2}$. If none of the $P^{-\mathfrak{G}_2}M_{\mathfrak{R}_2}$ are prime to $P$, $A_{\bar{R}_2}(D)\equiv0\ (\mathrm{mod}\,P^\lambda)$. Its value is also zero modulo $P^\lambda$ unless $r_1=\bar{r}_1$, $e_1=\bar{e}_1$ and $e_2=\bar{e}_2$ from the fact that each term carries an $A_{\bar{R}_1}$ function as factor and from the first factor. Thus $A_{\bar{R}_2}(D)$ is zero modulo $P^\lambda$ unless at least one of the $M^{(j)}_{\mathfrak{R}_2}$ is such that $P^{-\mathfrak{G}_2}M_{\mathfrak{R}_2}$ is prime to $P$ and at the same time contains a principal minor $M_{\mathfrak{R}_1}$ such that $P^{-\mathfrak{G}_1}M_{\mathfrak{R}_1}$ is prime to $P$. Let all these conditions be satisfied and take this $P^{-\mathfrak{G}_1}M_{\mathfrak{R}_1}=P^{-\bar{E}_1}M_{\bar{R}_1}$ as the $M_{R_1}$ of §§ 1 and 2. Then we have that $q_n$ is equivalent modulo $P^\lambda$ to a form whose determinant is $D^{(1)}$ of § 2. If $r_2<\bar{r}_2$, all the $\mathfrak{R}_2$-rowed minors of $D^{(1)}$ are divisible by a higher power of $P$ than $P^{\mathfrak{G}_2}$, and hence none of the $P^{-\mathfrak{G}_2}M_{\mathfrak{R}_2}$ of $D$ are prime to $P$. Therefore, $r_2\geqq\bar{r}_2$. If $r_2>\bar{r}_2$, at least one of the $R'_2$-rowed principal minors of $D^{(1)}$ that contains $M_{\mathfrak{R}_1}$ as its leading $R_1$-rowed principal minor is such that $P^{-E'_2}M_{R'_2}$ is prime to $P$, and $A_{\bar{R}_2}(D)$ is zero modulo $P^\lambda$ from its final factor by Fermat's Theorem. Hence $A_{\bar{R}_2}(D)\equiv0\ (\mathrm{mod}\,P^\lambda)$ unless $q_n$ belongs to a class $Ce_im_iR_k$ or $Ce_im_inD$ with $e_1=\bar{e}_1$, $e_2=\bar{e}_2$, $r_1=\bar{r}_1$, $r_2=\bar{r}_2$, and by (11) its value is 1 or $-1$ modulo $P^\lambda$ according as $m_2=0$ or 1. Similarly, we have that the function $(i=1,\ldots.,k-1)$

$$A_{\bar{R}_i}(D)\equiv(P^{e_i-\varepsilon_i})^\mu\{\prod_{j=1}^{i-1}A_{\bar{R}_j}(M^{(1)}_{\mathfrak{R}_i})\,(P^{-\mathfrak{G}_i}M^{(1)}_{\mathfrak{R}_i})^{\frac{\mu}{2}}$$
$$+\prod_{j=1}^{i-1}A_{\bar{R}_j}(M^{(2)}_{\mathfrak{R}_i})\,(P^{-\mathfrak{G}_2}M^{(2)}_{\mathfrak{R}_i})^{\frac{\mu}{2}}[1-(P^{-\mathfrak{G}_i}M^{(1)}_{\mathfrak{R}_i})^\mu]$$
$$+\ldots.+\prod_{j=1}^{i-1}A_{\bar{R}_j}(M^{(i)}_{\mathfrak{R}_i})\,(P^{-\mathfrak{G}_i}M^{(i)}_{\mathfrak{R}_i})^{\frac{\mu}{2}}\prod_{v=1}^{i-1}[1-(P^{-\mathfrak{G}_i}M^{(v)}_{\mathfrak{R}_i})^\mu]\}\prod_{w}[1-(P^{-E'_i}M^{(w)}_{R'_i})^\mu]$$
$$(\mathrm{mod}\,P^\lambda),\quad(14)$$

where $M^{(v)}_{\mathfrak{R}_i}$ ranges over the principal minors of $D$ of order $\mathfrak{R}_i$, and $M^{(w)}_{R'_i}$ ranges over the principal minors of $D$ of order $R'_i > \mathfrak{R}_i$, has the value zero unless $q_n$ belongs to a class $Ce_i m_i R_k$ or to a class $Ce_i m_j nD$ with $e_j = \bar{e}_j$, $r_j = \bar{r}_j$, and that for such forms its value is 1 or $-1$ modulo $P^\lambda$ according as $m_i = 0$ or 1. When $\bar{R}_k < n$, we have

$$\begin{aligned} A_{\bar{R}_k}(D) \equiv (P^{e_k - \bar{e}_k})^\mu \Big\{ & \prod_{i=1}^{k-1} A_{\bar{R}_i}(M^{(1)}_{\mathfrak{R}_k}) \, (P^{-\mathfrak{E}_k} M^{(1)}_{\mathfrak{R}_k})^{\frac{\mu}{2}} \\ & + \prod_{i=1}^{k-1} A_{\bar{R}_i}(M^{(2)}_{\mathfrak{R}_k}) \, (P^{-\mathfrak{E}_k} M^{(2)}_{\mathfrak{R}_k})^{\frac{\mu}{2}} [1 - (P^{-\mathfrak{E}_k} M^{(1)}_{\mathfrak{R}_k})^\mu] \\ & + \ldots + \prod_{i=1}^{k-1} A_{\bar{R}_i}(M^{(t)}_{\mathfrak{R}_k}) \, (P^{-\mathfrak{E}_k} M^{(t)}_{\mathfrak{R}_k})^{\frac{\mu}{2}} \prod_{j=1}^{t-1} [1 - (P^{-\mathfrak{E}_k} M^{(j)}_{\mathfrak{R}_k})^\mu] \Big\} \\ & \prod_v [1 - (P^{-E'_k} M^{(v)}_{R'_k})^\mu] \prod_w [1 - (P^{-(E_k + e_{k+1})} d^{(w)}_{R_k+1})^\mu] \pmod{P^\lambda}, \end{aligned} \tag{15}$$

where $M^{(j)}_{\mathfrak{R}_k}$ ranges over the principal minors of $D$ of order $\mathfrak{R}_k$, $M^{(v)}_{R'_k}$ ranges over those of order $R'_k > \mathfrak{R}_k$, $d^{(w)}_{R_k+1}$ ranges over the minors of $D$ of order $R_k + 1$, and $e_{k+1}$ is the $(R_k+1)$-th elementary exponent of $D$. In case $e_{k+1} \geqq \lambda$, define $P^{-(E_k + e_{k+1})} d^{(w)}_{R_k+1}$ to be zero modulo $P^\lambda$. Then $A_{\bar{R}_k}(D)$ has the value zero unless $e_i = \bar{e}_i$, $r_i = \bar{r}_i$ $(i = 1, \ldots, k)$ and $e_{k+1} \geqq \lambda$, whence $R_k = R_k = \bar{R}_k$. Hence $A_{\bar{R}_k}(D) \equiv 0 \pmod{P^\lambda}$ unless $q_n$ belongs to a class $Ce_i m_i R_k$ with $e_i = \bar{e}_i$ and $r_i = \bar{r}_i$ $(i = 1, \ldots, k)$, and for such forms its value is 1 or $-1$ according as $m_k = 0$ or 1. When $\bar{R}_k = n$ we have

$$\begin{aligned} A_n(D) \equiv (P^{e_k - \bar{e}_k})^\mu (P^{-E_k} D)^\mu \{1 - [(P^{-E_k + e_k} D - P^{-\bar{E}_k + \bar{e}_k} \bar{D}) P^{-q}]^\mu \} \Big\{ & \prod_{i=1}^{k-1} A_{\bar{R}_i}(D) \\ & \pmod{P^\lambda}, \end{aligned} \tag{16}$$

where $P^q$ is the highest power of $P$ that divides $P^{-E_k + e_k} D - P^{-\bar{E}_k + \bar{e}_k} \bar{D}$. In case $q \geqq \lambda$, define $P^{-q}(P^{-E_k + e_k} D - P^{-\bar{E}_k + \bar{e}_k} \bar{D})$ to be zero modulo $P^\lambda$. Then the value of $A_n(D)$ is zero modulo $P^\lambda$ unless $e_i = \bar{e}_i$, $r_i = \bar{r}_i$ $(i = 1, \ldots, k)$ and $P^{-E_k + e_k} D \equiv P^{-\bar{E}_k + \bar{e}_k} \bar{D} \pmod{P^\lambda}$. The function is also zero unless $P^{-E_k} D$ is prime to $P$, *i. e.*, unless the rank modulo $P^\lambda$ of $D$ is $R_k$. Hence $A_n(D)$ is zero modulo $P^\lambda$ unless $q_n$ belongs to a class $Ce_i m_j nD$ with $e_i = \bar{e}_i$, $r_i = \bar{r}_i$ $(i = 1, \ldots, k)$ and its value is 1 modulo $P^\lambda$ for all such forms.

Finally, define $P^{-e_1} a_{ij}$ to be zero modulo $P^\lambda$ when $e_1 \geqq \lambda$. Then

$$I \equiv \Pi [1 - (P^{-e_1} a_{ij})^\mu] \pmod{P^\lambda} \qquad (i, j = 1, \ldots, n;\ i \leqq j) \tag{17}$$

has the value zero modulo $P^\lambda$ unless $q_n$ belongs to the class $C_0$ and the value 1 for such a form. The functions $A_{\bar{R}_i}(D)$, $A_n(D)$ and $I$ are modular invariants that completely characterize the classes of modular quadratic forms modulo $P^\lambda$.

5. *Characteristic Modular Invariants.*—An invariant $\phi$ that has the value 1 for all forms of one class, and the value zero for forms of any other

class is called a characteristic invariant. In case such a $\phi$ is a modular invariant it is called a characteristic modular invariant. For example, $I$ is a characteristic invariant for the class $C_0$.

Let $\bar{e}_i, \bar{r}_i, \overline{m}_i$ $(i=1, \ldots, k)$ be a set of consistent values that define the class $C\bar{e}_i\overline{m}_i\bar{R}_k$. Among the integers $1, \ldots, k$ a certain number $s_1$ of them will correspond to $m$'s that are zero. The remaining $s_2$ of them will correspond to $m$'s that are unity. Call the first set $\sigma_1$, and the second, $\sigma_2$, and consider the function

$$A_{\bar{R}_k\sigma_1\sigma_2} \equiv \frac{1}{2^{s_1+s_2}} \prod_i [(A_{\bar{R}_i}(D))^2 + A_{\bar{R}_i}(D)] \prod_j [(A_{\bar{R}_j}(D))^2 - A_{\bar{R}_j}(D)] \pmod{P^\lambda}, \quad (18)$$

where $i$ ranges over $\sigma_1$, and $j$ ranges over $\sigma_2$. This function has the value 1 * for all forms of class $C\bar{e}_i\overline{m}_i\bar{R}_k$ and the value zero for all other forms. Hence $A_{\bar{R}_k\sigma_1\sigma_2}$ is a characteristic modular invariant of $q_n$.

Next let $\bar{e}_i, \bar{r}_i$ $(i=1, \ldots, k)$, $\overline{m}_j$ $(j=1, \ldots, k-1)$ and $\overline{D}$ be a consistent set of values of these quantities that define the class $C\bar{e}_i\overline{m}_jn\overline{D}$; thus $\bar{R}_k=n$. Separate the integers $1, \ldots, k-1$ into sets $\sigma_1$ and $\sigma_2$ composed of $s_1$ and $s_2$ elements, respectively, such that those of $\sigma_1$ correspond to the $m$'s that are zero, while those of $\sigma_2$ correspond to the $m$'s that are unity. Then the function

$$A_{n\sigma_1\sigma_2} \equiv \frac{1}{2^{s_1+s_2}} \prod_i [(A_{\bar{R}_i}(D))^2 + A_{\bar{R}_i}(D)] \prod_j [(A_{\bar{R}_j}(D))^2 - A_{\bar{R}_j}(D)] A_n(D) \pmod{P}, \quad (19)$$

where $i$ ranges over $\sigma_1$ and $j$ ranges over $\sigma_2$, has the value 1 for all forms of class $C\bar{e}_i\overline{m}_jn\overline{D}$ and the value zero for all other forms.

Hence the characteristic invariants $A_{\bar{R}_k\sigma_1\sigma_2}$, $A_{n\sigma_1\sigma_2}$, and $I$ completely characterize the classes of modular quadratic forms modulo $P^\lambda$.

For the case $\lambda=1$, $A_{\bar{R}_0}(D)$ and $A_{\bar{R}_k\sigma_1\sigma_2}$ become, respectively, the invariants $A_r$ and $I_{r,\pm 1}$ given by Professor Dickson in his Madison Colloquium lectures, pp. 11, 13. The function $A_{n\sigma_1\sigma_2}$ is for this case a characteristic invariant of his class $C_{nD}$, while $I$ in this case is replaced by the simple form $I=\Pi(1-a_{ij}{}^\mu)$, as given *loc. cit.*, p. 11. These are the only ones that enter in this case, since the only set of integers $e_i, m_i, r_i$ are $0, r_1, m_1$. Thus, for this case the invariants given above are all polynomial modular invariants. This may also happen for the case $\lambda>1$, but would obviously be a very special case.

6. *Number of Linearly Independent Modular Invariants of $q_n$.*—For convenience denote the classes of modular quadratic forms modulo $P^\lambda$ by $C_0, C_1, C_2, \ldots$ and the corresponding characteristic invariants by

* In case either $\sigma_1$ or $\sigma_2$ is an empty set, the corresponding product factors of (18) is defined to be unity. Similarly in (19).

$I_0, I_1, I_2, \cdots$. Since any modular invariant $I$ of $q_n$ takes certain values $v_0, v_1, v_2, \ldots$ for the respective classes $C_0, C_1, C_2, \ldots$ we have

$$I \equiv v_0 I_0 + v_1 I_1 + v_2 I_2 + \ldots \pmod{P^\lambda}.$$

Hence, any modular invariant is congruent to a linear homogeneous function of the characteristic invariants. Moreover, the number of linearly independent modular invariants of the quadratic form $q_n$ modulo $P^\lambda$ equals the number of classes.

## II. Polynomial Modular Invariants of a Binary Quadratic Form.

7. *Invariants for a Prime Power Modulus $P^\lambda$.*—Consider the form

$$f = a_0 x^2 + a_1 xy + a_2 y^2, \tag{1}$$

where the $a_i$ are integers taken modulo $P^\lambda$. The transformations *

$$\begin{aligned} R&: x = x' + Ty', \quad y = y', \\ S&: x = x', \quad y = ky', \quad (k \text{ prime to } P) \\ Q&: x = y', \quad y = -x', \end{aligned}$$

generate the group of all linear homogeneous transformations whose determinant is prime to $P$ and coefficients are integers modulo $P^\lambda$. Transformations $R, S, Q$ give the respective replacements

$$a_0' = a_0, \quad a_1' = a_1 + 2a_0 T, \quad a_2' = a_2 + a_1 T + a_0 T^2. \tag{2}$$

$$a_0' = a_0, \quad a_1' = k a_1, \quad a_2' = k^2 a_2. \tag{3}$$

$$a_0' = a_2, \quad a_1' = -a_1, \quad a_2' = a_0. \tag{4}$$

Theorem 1. *The form $f$ has the absolute invariant modulo $P^\lambda$*

$$H = \prod_{i=0}^{2} (a_i^\mu - 1) \qquad \mu = \phi(P^\lambda) = P^{\lambda-1}(P-1).$$

Evidently replacements (3) and (4) leave $H$ unaltered. It remains to show that $H$ is unaltered by replacement (2). This is evident if $a_0$ is prime to $P$, since then $a_0^\mu - 1 \equiv 0 \pmod{P^\lambda}$. Hence let $a_0$ be divisible by $P$. Then $a_1'$ is of the form $a_1 + Pq$ and, by induction on $\lambda$,

$$(a_1 + Pq)^{P^{\lambda-1}} \equiv a_1^{P^{\lambda-1}} \pmod{P^\lambda},$$

whence $a_1'^\mu \equiv a_1^\mu$. Hence $H' \equiv A(a_2'^\mu - 1)$, where

$$A = (a_0^\mu - 1)(a_1^\mu - 1).$$

Thus $H' \equiv H$ if either $a_0$ or $a_1$ is prime to $P$. In the contrary case, $a_2 = a_2 + Pq$ and $a_2'^\mu \equiv a_2^\mu$ as before, whence $H' \equiv H$ in all cases.

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* Cf. C. Jordan, "Traité des Substitutions," p. 93.

The same method leads to

THEOREM 2. *The form f has the absolute invariants modulo* $2^\lambda$

$$I=a_1^\mu=a_1^{2^{\lambda-1}}, \quad Q=a_0^\mu a_1^\mu a_2^\mu.$$

8. *Fundamental System of Invariants Modulo* 4.

THEOREM 3. *Necessary and sufficient conditions that*

$$\alpha+\beta T+\gamma T^2+\delta T^3\equiv 0 \pmod 4$$

*for all integral values of* $T$ *are*

$$\alpha\equiv 0, \quad 2\beta\equiv 2\gamma\equiv\beta+\gamma+\delta\equiv 0 \pmod 4.$$

The general invariants of § 7 become

$$H=\prod_{i=0}^{2}(a_i^3-1), \quad I=a_1^3, \quad Q=a_0^3a_1^3a_2^3.$$

A useful combination of these invariants is

$$S=P-Q-I+1=a_0^2+a_2^2-a_0^2a_2^2-a_0^2a_1^2-a_1^2a_2^2.$$

THEOREM 4. *The form f has the absolute invariants* *

$$\begin{aligned}
K&\equiv a_1+a_1^3+2a_0a_2,\\
J&\equiv 2a_0+2a_2+a_0a_1+a_0a_1^2+a_1a_2+a_1a_2^2+a_0a_1a_2+a_0a_1^2a_2,\\
E&\equiv a_0a_2+a_0^2a_2^2+a_0a_2(a_1+a_1^2+a_1^3)\\
&\equiv a_0^2a_2+a_0a_2^2+a_0^2a_2^2-a_0^2a_2^2+a_0a_1a_2^2+a_0^2a_1^2a_2+a_0^2a_1^2a_2^2 \pmod 4.
\end{aligned}$$

The following identical congruences are of use in simplifying the discussion given below for the general invariant.

$$U\equiv(a_1+a_1^2)(a_2+a_2^2)\equiv 0, \quad V\equiv(a_0+a_0^2)(a_1+a_1^2)\equiv 0, \quad L\equiv(a_0+a_0^2)(a_2+a_2^2)\equiv 0,$$

$$Q_1=3Q\equiv a_0^2a_1^2a_2^2+a_0^2a_1^2a_2^2+a_0^2a_1^2a_2^2, \quad Q_2=3Q\equiv a_0^2a_1^2a_2+a_0^2a_1^2a_2+a_0^2a_1^2a_2^2,$$

$$M\equiv a_0a_1a_2^2+a_0^2a_1a_2^2+a_0a_1^2a_2+a_0^2a_1^2a_2\equiv 4Q\equiv 0,$$

$$N\equiv a_0a_1a_2^2+a_0^2a_1a_2^2+a_0a_1^2a_2+a_0^2a_1^2a_2\equiv 4Q\equiv 0.$$

In view of the identical congruence $x^4\equiv x^2 \pmod 4$, we may assume that the exponent of each $a_i$ in an invariant of $f$ does not exceed 3. Hence, we may denote any invariant $\phi$ by

$$\phi\equiv\sum_{i,j=0}^{3}A_{ij}a_1^ia_2^j, \quad A_{ij}=\alpha_{ij}+\beta_{ij}a_0+\gamma_{ij}a_0^2+\delta_{ij}a_0^3, \tag{5}$$

where $\alpha_{ij}$, etc., are constants. The difference

$$\phi-(A_{11}+A_{12}a_2+A_{31}a_1+A_{33}a_1a_2)U$$

has

$$A_{11}\equiv A_{12}\equiv A_{31}\equiv A_{33}\equiv 0.$$

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* $K$, $J$ and $E$ were discovered by carrying through for special cases the discussion made below for finding the general invariant; they were verified to be invariant with respect to the generators $R, S, Q$.

Hence, we may assume that in the initial $\phi$ these quantities are zero modulo 4. Similarly, by considering in turn

$$\phi-[\beta_{01}+(\gamma_{01}-\beta_{01})a_0]L-(\delta_{01}-\gamma_{01}+\beta_{01})E,\quad \phi-\alpha_{10}K-[\beta_{10}+(\gamma_{10}-\beta_{10})a_0]V,$$
$$\phi-[\beta_{30}+(\gamma_{30}-\beta_{30})a_0]a_1V,\qquad \phi-[\beta_{03}+(\gamma_{03}-\beta_{03})a_0]a_2L,$$

we may assume that

$$A_{01}\equiv\alpha_{01},\quad A_{10}\equiv\delta_{10}a_0^3,\quad A_{30}\equiv\alpha_{30}+\delta_{30}a_0^3,\quad A_{03}\equiv\alpha_{03}+\delta_{03}a_0^3.$$

Then by employing in turn

$$\phi-\beta_{21}M-\gamma_{21}Q_2,\quad \phi-\beta_{32}N-\gamma_{32}Q_1,\quad \phi-\alpha_{12}J-\gamma_{12}Q_2,\quad \phi-\alpha_{20}I-\gamma_{22}Q+\alpha_{22}S,$$

we may take

$$A_{21}\equiv\alpha_{21}+\delta_{21}a_0^3,\quad A_{32}\equiv\alpha_{32}+\delta_{32}a_0^3,\quad A_{12}\equiv\beta_{12}a_0+\delta_{12}a_0^3,\quad \alpha_{20}=\alpha_{22}=\gamma_{22}=0.$$

At no step have any of the quantities previously taken to be zero been disturbed.

Subjecting the simplified $\phi$ to replacement (2) we get

$$\phi'\equiv\phi+\beta T+\gamma T^2+\delta T^3,$$

where

$$\begin{aligned}
\beta\equiv{}&2A_{10}a_0+2A_{12}a_0+2A_{02}a_0a_2+2A_{12}a_0a_2+(A_{01}+2A_{02}a_0+2A_{22}a_0+2A_{30}a_0\\
&+2A_{32}a_0+2A_{02}a_2+2A_{03}a_0a_2+2A_{12}a_2+2A_{12}a_0a_2+2A_{22}a_2+2A_{22}a_0a_2\\
&+2A_{23}a_0a_2+2A_{32}a_2+3A_{03}a_2^2)a_1+(A_{21}+3A_{23}a_2^2)a_1^3,\\
\gamma\equiv{}&A_{01}a_0+A_{02}a_0^2+A_{03}a_0^3+3A_{03}a_0a_2^2+3A_{03}a_0^2a_2+A_{12}a_0^2a_1+(A_{02}+3A_{03}a_2+A_{21}a_0\\
&+3A_{03}a_0+A_{23}+A_{22}a_0^2+3A_{22}a_0a_2^2+3A_{23}a_2+3A_{23}a_0^2a_2+3A_{23}a_0+A_{23}a_0^3)a_1^2\\
&+(A_{12}+A_{32}+A_{32}a_0^2)a_1^3,\\
\delta\equiv{}&3A_{03}a_0^2a_1+(A_{03}+A_{23}+3A_{23}a_0^2)a_1^3.
\end{aligned}$$

By Theorem 3, necessary and sufficient conditions that $\phi'-\phi\equiv0$ for all integral values of $T$ are $2\beta\equiv2\gamma\equiv\beta+\gamma+\delta\equiv0$. From the form of $A_{10}$, $A_{12}$ and $A_{22}$ we see that

$$2(A_{12}+A_{12}a_0)\equiv0,\quad 2(A_{22}+A_{22}a_0)\equiv0,\quad 2(A_{10}+A_{10}a_0)\equiv0. \tag{6}$$

From $2\beta\equiv2(A_{01}+A_{21})a_1+2(A_{03}+A_{23})a_1a_2$, we have

$$2(A_{01}+A_{21})\equiv0,\quad 2(A_{03}+A_{23})\equiv0. \tag{7}$$

Next,

$$2\gamma\equiv2(A_{01}+A_{02}+A_{03})a_0+2[A_{02}+A_{22}+A_{12}+A_{32}+a_0(A_{12}+A_{21}+A_{03}+A_{22}+A_{32})]a_1.$$

Hence, in view of (7), and (6),

$$2(A_{01}+A_{02}+A_{03})a_0\equiv0,\quad 2(A_{02}+A_{32})(1+a_0)\equiv0. \tag{8}$$